Fourier transforms in the solution of the driven harmonic oscillator

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1 Introduction

Much of the complex analysis used in the solution of quantum scattering problems as well as classical electromagnetic radiation and scattering can be developed for the related problem of the driven classical harmonic oscillator.

2 Fourier Transforms

The Fourier transform is most easily developed from the limit of a Fourier series. The Fourier series basis functions are sines, cosines, or complex exponentials. They are the solution of the Sturm-Liouville eigenvalue differential equation

$$-\frac{d^2}{dx^2}\psi_n(x) = k_n^2\psi_n(x) \tag{1}$$

along with boundary conditions. Particularly for quantum problems, but also for others, the equations simplify if the basis functions are eigenfunctions of the derivative d/dx. That is we would like to use the complex exponentials for the eigenfunctions. Periodic boundary conditions, where $-L/2 \leq x \leq L/2$, with the functions and their derivatives matched at $\pm L/2$ leads to the solutions

$$\psi_n(x) = e^{ik_n x} \tag{2}$$

with $k_n = 2\pi n/L$, and n an integer. The different $\psi_n(x)$ are orthogonal. Since these functions are complete we can expand any function in the range $-L/2 < x < L/2$ as

$$f(x) = \sum_{n=-\infty}^{\infty} e^{ik_n x}\tilde{f}_n \tag{3}$$

where $\tilde{f}_n$ are the expansion coefficients. Multiplying by $e^{-ik_m x}$ and integrating, we have

$$\int_{-L/2}^{L/2} dx e^{-ik_m x} f(x) = \int_{-L/2}^{L/2} dx e^{-ik_m x} \sum_{n=-\infty}^{\infty} e^{ik_n x}\tilde{f}_n = L\tilde{f}_m \equiv C\tilde{f}(k_n)\,, \tag{4}$$

with C an arbitrary constant.

We now take the limit that $L \to \infty$. Since physical functions are not periodic with a period $L \to \infty$, this only makes sense for functions that go to a constant at large $|x|$. Typically, the Fourier transform is only useful if the functions are localized and go to zero for large $|x|$. For $L \to \infty$, the spacing between the k_n goes to zero, and the sum over n can be replaced by an integral over n,

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} e^{ik_n x}\tilde{f}_n = \int_{-\infty}^{\infty} dn e^{ik_n x}\frac{C}{L}\tilde{f}(k_n) \\ &= C\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\tilde{f}(k)\,. \end{aligned} \tag{5}$$

The transform pair are then

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{C}\int_{-\infty}^{\infty} dx e^{-ikx} f(x) \\ f(x) &= C\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\tilde{f}(k)\,. \end{aligned} \tag{6}$$

The most commonly used conventions for the transform pair are $C = 1$, which puts the $(2\pi)^{-1}$ factor on the k integral, $C = \sqrt{2\pi}$, which puts $(2\pi)^{-1/2}$ symmetrically on both integrals, and $C = 2\pi$ which puts the $(2\pi)^{-1}$ factor on the x integral. I nearly always put the 2π factors on the k integral, for a k - x pair, or the ω integral for an ω - t pair. This agrees with the normalizations for the position and momentum eigenstates of $\langle x|x'\rangle = \delta(x - x')$ with $\langle x|p\rangle = e^{\frac{i}{\hbar}px}$, and $\langle p|p'\rangle = 2\pi\hbar\delta(p - p')$.

3 Solution of the driven harmonic oscillator by Fourier transforms

Newton's equation for the position $x(t)$ of a driven undamped harmonic oscillator of mass m and spring constant $m\omega_0^2$ is

$$m\frac{d^2x(t)}{dt^2} = -m\omega_0^2 x(t) + F(t) \tag{7}$$

where $F(t)$ is the driving force at time t.

For the Fourier transform to be well defined, we require that the function to be transformed go to zero at $t \to \pm\infty$. We therefore imagine the force is applied for a finite amount of time. However, we know that if we excite the harmonic oscillator and thereafter have the driving force zero, the undamped oscillator will continue to oscillate forever, and the Fourier transform of $x(t)$ will not be well defined. The standard method of handling this is to modify the equation of motion, usually to include a small amount of damping, so that $x(t)$ goes to zero. Here we will examine the analytic properties of the Fourier transformed solution to obtain the required result.

Fourier transforming the equation of motion and integrating the acceleration term by parts twice gives

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} dt e^{i\omega t}\left[m\frac{d^2x(t)}{dt^2} + m\omega_0^2 x(t) - F(t)\right] \\ &= \int_{-\infty}^{\infty} dt e^{i\omega t} m\left\{x(t)[\omega_0^2 - \omega^2] - F(t)\right\} \\ &= \tilde{x}(\omega)m[\omega_0^2 - \omega^2] - \tilde{F}(\omega) \end{aligned} \tag{8}$$

where we assume that $x(t)$ and $dx(t)/dt$ are zero at $t \to \pm\infty$ so that the surface terms can be dropped. Solving

$$\tilde{x}(\omega) = \frac{\tilde{F}(\omega)}{m(\omega_0^2 - \omega^2)} . \tag{9}$$

Notice this diverges at $\omega = \pm\omega_0$ reflecting the problem with defining the Fourier transform. To see how to modify the equations around $\omega = \pm\omega_0$, we Fourier transform back to $x(t)$ to give

$$x(t) \stackrel{?}{=} \frac{1}{m}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{\tilde{F}(\omega)}{\omega_0^2 - \omega^2} . \tag{10}$$

where the question mark indicates we have yet to deal with the divergent integrand. Notice that integration around the poles at $\omega = \pm\omega_0$ will give contributions proportional to $e^{\pm i\omega_0 t}$ which are the homogeneous solutions to the differential equations. Modifying how we handle the poles corresponds to selecting the amounts of each of these solutions to add, and sets the boundary conditions.

If the oscillator is at rest at the origin before the driving force is applied, the boundary condition is $x(t \to -\infty) = 0$. We look at our integral solution at large negative t and note that we can add to the integral along the real ω axis, an integral around the large semicircle in the upper half plane, since the factor $e^{-i\omega t}$ will make that integral zero. To enforce our boundary condition the contour integration around this semicircle and the real ω axis must go to zero for all large negative times. Notice that any poles in the upper half plane with a finite imaginary part will give residues that are damped exponentially at large negative t. However, poles infinitesimally close to the real axis will not be damped. To satisfy the boundary condition we move the singularities at $\omega = \pm\omega_0$ slightly into the lower half plane so their residues cannot contribute at large negative times. We get the solution

$$x(t) = \frac{1}{m}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{\tilde{F}(\omega)}{\omega_0^2 - (\omega + i\eta)^2} , \tag{11}$$

where η is a positive infinitesimal. The Fourier transform is

$$\tilde{x}(\omega) = \frac{1}{m}\frac{\tilde{F}(\omega)}{\omega_0^2 - (\omega + i\eta)^2} . \tag{12}$$

An alternative way to get to this equation is to change the equation of motion so that the function $x(t)$ can be Fourier transformed without difficulty. If we had instead solved the problem of a harmonic oscillator initially at rest at the origin with a frictional damping force $-m\gamma v(t)$, the boundary condition enforces $x(t \to -\infty) = 0$, and the dampling force ensures that $x(t \to \infty) = 0$. Therefore this $x(t)$ can be transformed immediately. Setting $\gamma = \eta$, a positive infinitesimal, at the end of the calculation, gives Eq. 11.

4 Energy transfered to the harmonic oscillator

Let's look at the solution at $t \to \infty$. From the argument above, only the poles infinitesimally close to the real axis will survive. We close in the lower half plane, and the residues give

$$x(t \to \infty) = \frac{i}{2m\omega_0}\left[\tilde{F}(-\omega_0)^{i\omega_0 t} - \tilde{F}(\omega_0)^{-i\omega_0 t}\right] . \tag{13}$$

Since $F(t)$ is real, $F(-\omega) = F^*(\omega)$, and we have

$$x(t \to \infty) = \frac{1}{m\omega_0}\mathrm{Im}\left[\tilde{F}(\omega_0)e^{-i\omega t}\right] = \frac{1}{m\omega_0}\left[-\mathrm{Re}F(\omega_0)\sin(\omega t) + \mathrm{Im}F(\omega_0)\cos(\omega t)\right]. \tag{14}$$

The energy of the harmonic oscillator evaluated from $E = \frac{1}{2}mv(t)^2 + \frac{1}{2}m\omega_0^2 x(t)^2$ at long times is then

$$E = \frac{1}{2m}|\tilde{F}(\omega_0)|^2 . \tag{15}$$

This should equal the work done by the force which is

$$\begin{aligned} W &= \int_{-\infty}^{\infty} dt v(t)F(t) \\ &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\tilde{v}^*(\omega)e^{i\omega t}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\tilde{F}(\omega)e^{-i\omega' t} \\ &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{v}^*(\omega)\tilde{F}(\omega) = \frac{1}{\pi}\mathrm{Re}\int_0^{\infty} d\omega\tilde{v}^*(\omega)\tilde{F}(\omega) \end{aligned} \tag{16}$$

The Fourier transform of the velocity is

$$\tilde{v}(\omega) = -i\omega\tilde{x}(\omega) = \frac{-i\omega}{m}\frac{\tilde{F}(\omega)}{\omega_0^2 - (\omega + i\eta)^2} . \tag{17}$$

Plugging this in, we have

$$W = \mathrm{Im}\int_0^{\infty} d\omega\frac{\omega}{m\pi}\frac{|\tilde{F}(\omega)|^2}{\omega_0^2 - (\omega + i\eta)^2} . \tag{18}$$

We can make the formal replacement under an integral

$$\frac{1}{x \pm i\eta} = \mathrm{P}\frac{1}{x} \mp i\pi\delta(x) , \tag{19}$$

where P indicates the principal parts integration.[1]

Our result is then

$$W = \frac{1}{2m}|\tilde{F}(\omega_0)|^2 . \tag{22}$$

[1]To show this we can require that the function above is multiplied by a function analytic in the region around $x = 0$. Since the integrand is analytic, we can then distort the contour into a small semicircle at $x = 0$, where the semicircle is in the lower half plane for the $-i\eta$ term, and the semicircle in the upper half plane for the $i\eta$ term, since these distortions will not cause the contour to cross the pole at $\pm i\eta$. Now take the limit that η goes to zero, and then the limit that the semicircle radius shrinks to zero. The portion of the contour along the real axis gives the principal parts integral, while the semicircle gives $\mp i\pi$ times the residue, and the residue is picked up by the delta function. Alternatively, without using contour integration, divide the integral into a principal parts integration and the integration on the segment between the principal parts limits. The integral becomes

$$\int_{-\epsilon}^{\epsilon} f(x)\frac{1}{x \pm i\eta} = \int_{-\epsilon}^{\epsilon} dx f(x)\frac{x \mp i\eta}{x^2 + \eta^2} . \tag{20}$$

For small enough ϵ, a well behaved $f(x)$, can be replaced by $f(0)$, the x term in the numerator gives zero. Changing variables to $u = x/\eta$ and taking the limit $\eta \to 0$ this becomes

$$\int_{-\epsilon}^{\epsilon} f(x)\frac{1}{x \pm i\eta} = f(0)\int_{-\infty}^{\infty} du\frac{\mp i}{u^2 + 1} = \mp i\pi f(0) , \tag{21}$$

and the formal replacement is again shown.

5 Green's function Equation

We can evaluate Eq. 11 as

$$\begin{aligned} x(t) &= \frac{1}{m}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\frac{\tilde{F}(\omega)}{\omega_0^2-(\omega+i\eta)^2} \\ &= \frac{1}{m}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\frac{1}{\omega_0^2-(\omega+i\eta)^2}\int_{-\infty}^{\infty}dt'e^{i\omega t'}F(t') \\ &= \int_{-\infty}^{\infty}dt'F(t')\underbrace{\frac{1}{m}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega_0^2-(\omega+i\eta)^2}}_{\equiv G(t-t')}\,. \end{aligned} \tag{23}$$

For $t < t'$, we can close the contour for the Green's function integral with a large semicircle in the upper half plane. The poles are in the lower half plane, so the integral gives zero. For $t > t'$, we close with a large semicircle in the lower half plane. The poles are at $\pm\omega - i\eta$, and the residues after taking $\eta \to 0$ are $e^{-i(\pm\omega_0)(t-t')}/(\mp 4\pi m\omega_0)$. The integral is $-2\pi i$ times the sum of the residues giving

$$G(t-t') = \begin{cases} 0 & t < t' \\ \frac{1}{m\omega_0}\sin[\omega_0(t-t')] & t > t' \end{cases}. \tag{24}$$

Notice that we could write this retarded Green's function almost immediately since it is the solution for $x(t)$ initially at rest with a unit impulsive force acting at $t = t'$. Before the impulse the solution is 0. After the impulse, the solution must be a free oscillator solution and be continuous at $t = t'$, i.e. $A\sin[\omega_0(t-t')]$. The unit impulse gives a unit momentum just after it is applied. Taking the derivative at $t = t'^{+}$, gives a velocity $A\omega_0$, so $mA\omega_0 = 1$, and $A = 1/m\omega_0$.

6 Application to Classical Radiation

The power of these methods can be shown by looking at the problem of classical radiation from a charge and current distribution. We imagine that the charge and current densities are localized in space and time, so they can be Fourier transformed. In Lorentz gauge, Maxwell's equations are

$$\begin{aligned} \nabla\cdot\mathbf{A}(\mathbf{r},t)+\frac{1}{c}\frac{\partial}{\partial t}\Phi(\mathbf{r},t) &= 0 \\ \left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}(\mathbf{r},t) &= -\frac{4\pi}{c}\mathbf{J}(\mathbf{r},t) \\ \left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi(\mathbf{r},t) &= -4\pi\rho(\mathbf{r},t)\,. \\ \mathbf{E}(\mathbf{r},t) &= -\nabla\Phi(\mathbf{r},t)-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) \\ \mathbf{B}(\mathbf{r},t) &= \nabla\times\mathbf{A}(\mathbf{r},t) \end{aligned} \tag{25}$$

Fourier transforming by multiplying by $e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}$ and integrating over space and time gives the result

$$(\omega^2-k^2c^2)\tilde{\mathbf{A}}(\mathbf{k},\omega) = -4\pi c\tilde{\mathbf{J}}(\mathbf{k},\omega)$$

$$
\begin{aligned}
\omega\tilde{\Phi}(\mathbf{k},\omega) &= \mathbf{k}\cdot\tilde{\mathbf{A}}(\mathbf{k},\omega) \\
\tilde{\mathbf{E}}(\mathbf{k},\omega) &= -i\mathbf{k}\tilde{\Phi}(\mathbf{k},\omega)+\frac{i\omega}{c}\tilde{\mathbf{A}}(\mathbf{k},\omega) \\
\tilde{\mathbf{B}}(\mathbf{k},\omega) &= i\mathbf{k}\times\tilde{\mathbf{A}}(\mathbf{k},\omega)\,.
\end{aligned} \tag{26}
$$

The first of these equations is identical in form to the driven harmonic oscillator. Therefore, to match the boundary condition that the fields are zero at $t \to -\infty$, we handle the poles exactly as for the harmonic oscillator, with solution

$$
\tilde{\mathbf{A}}(\mathbf{k},\omega) = \frac{4\pi c}{k^2c^2-(\omega+i\eta)^2}\tilde{\mathbf{J}}(\mathbf{k},\omega) \tag{27}
$$

so that

$$
\begin{aligned}
\tilde{\mathbf{B}}(\mathbf{k},\omega) &= -\frac{i4\pi c}{k^2c^2-(\omega+i\eta)^2}\mathbf{k}\times\tilde{\mathbf{J}}(\mathbf{k},\omega) \\
\tilde{\mathbf{E}}(\mathbf{k},\omega) &= \frac{4\pi i}{\omega}\frac{\omega^2\tilde{\mathbf{J}}(\mathbf{k},\omega)-c^2\mathbf{k}\mathbf{k}\cdot\tilde{\mathbf{J}}(\mathbf{k},\omega)}{k^2c^2-(\omega+i\eta)^2}\,.
\end{aligned} \tag{28}
$$

The work done by the charges on the electromagnetic field must be the negative of the work done on the charges by the field or

$$
I = -\int_{-\infty}^{\infty} dt \sum_i \mathbf{v}_i(t)\cdot\mathbf{F}_i(t)\,. \tag{29}
$$

Plugging in the result of the Lorentz force, $F_i(t) = q_i\left[\mathbf{E}(\mathbf{r}_i(t),t)+\frac{\mathbf{v}_i(t)}{c}\times\mathbf{B}(\mathbf{r}_i(t),t)\right]$, this becomes

$$
I = -\int_{-\infty}^{\infty} dt \sum_i q_i\mathbf{v}_i(t)\cdot\mathbf{E}(\mathbf{r}_i(t),t)\,. \tag{30}
$$

Using the current density for a set of charges $\mathbf{J}(\mathbf{r},t) = \sum_i q_i\mathbf{v}_i(t)\delta[\mathbf{r}-\mathbf{r}_i(t)]$, this becomes just as in the energy expression for the harmonic oscillator

$$
\begin{aligned}
I &= -\int d^3r\int dt\mathbf{J}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t) = -\mathrm{Re}\int_0^\infty\frac{d\omega}{\pi}\int\frac{d^3k}{(2\pi)^3}\tilde{\mathbf{J}}^*(\mathbf{k},\omega)\cdot\tilde{\mathbf{E}}(\mathbf{k},\omega) \\
&= -\mathrm{Re}\int_0^\infty\frac{d\omega}{\pi}\int\frac{d^3k}{(2\pi)^3}\frac{4\pi i}{\omega}\frac{\omega^2\left|\tilde{\mathbf{J}}(\mathbf{k},\omega)\right|^2-c^2\left|\mathbf{k}\cdot\tilde{\mathbf{J}}(\mathbf{k},\omega)\right|^2}{k^2c^2-(\omega+i\eta)^2}\,.
\end{aligned} \tag{31}
$$

As in the harmonic oscillator case, only the delta function from the $i\eta$ gives a real result. The delta function makes the length $kc = \omega$, and only the transverse part of the current contributes. We get

$$
\begin{aligned}
I &= \int_0^\infty\frac{d\omega}{\pi}\int\frac{d^3k}{(2\pi)^3}\frac{4\pi}{\omega}\left[\omega^2\left|\tilde{\mathbf{J}}(\mathbf{k},\omega)\right|^2-c^2\left|\mathbf{k}\cdot\tilde{\mathbf{J}}(\mathbf{k},\omega)\right|^2\right]\pi\delta(k^2c^2-\omega^2) \\
&= \int_0^\infty d\omega\int d\Omega\frac{\omega^2}{4\pi^2c^3}\left|\hat{\mathbf{k}}\times\hat{\mathbf{k}}\times\tilde{\mathbf{J}}\left(\hat{\mathbf{k}}\frac{\omega}{c},\omega\right)\right|^2
\end{aligned} \tag{32}
$$

where $d\Omega = d\cos\theta d\phi$, and $\hat{\mathbf{k}} = \hat{\mathbf{x}}\sin\theta\cos\phi+\hat{\mathbf{y}}\sin\theta\sin\phi+\hat{\mathbf{z}}\cos\theta$. The intensity distribution in frequency and angle is

$$
\frac{d^2I}{d\Omega d\omega} = \frac{\omega^2}{4\pi^2c^3}\left|\hat{\mathbf{k}}\times\hat{\mathbf{k}}\times\tilde{\mathbf{J}}\left(\hat{\mathbf{k}}\frac{\omega}{c},\omega\right)\right|^2\,. \tag{33}
$$

7 Green's function for the wave equation

The Green's function for the wave equation can be calculated as for the harmonic oscillator,

$$A(\mathbf{r},t) = \int d^3r dt G(\mathbf{r},t,\mathbf{r}',t')\mathbf{J}(\mathbf{r}',t') \tag{34}$$

with the simplifying definitions $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $T = t - t'$,

$$G(\mathbf{r},t,\mathbf{r}',t') = \int \frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi}e^{i\mathbf{k}\cdot\mathbf{R}}e^{-i\omega T}\frac{4\pi c}{k^2c^2 - (\omega + i\eta)^2} \tag{35}$$

The ω integral is identical to the one done for the harmonic oscillator with the result

$$\begin{aligned}
G(\mathbf{r},t,\mathbf{r}',t') &= \int \frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{R}}\frac{4\pi}{k}\sin(kcT)\Theta(T) \\
&= \frac{2}{\pi R}\Theta(T)\int_0^\infty dk \sin(kR)\sin(kcT) \\
&= \frac{1}{\pi R}\Theta(T)\int_{-\infty}^\infty dk \sin(kR)\sin(kcT) \\
&= \frac{1}{2\pi R}\Theta(T)\int_{-\infty}^\infty dk \left\{\cos[k(R-cT)] - \cos[k(R+cT)]\right\} \\
&= \frac{1}{R}\Theta(T)\left[\delta(R-cT) - \delta(R+cT)\right] .
\end{aligned} \tag{36}$$

Since $R > 0$ and $\Theta(T)$ is zero for $T < 0$, the second delta function never contributes. The result is

$$G(\mathbf{r},t,\mathbf{r}',t') = \frac{1}{R}\Theta(T)\delta(R - cT) \tag{37}$$

which is retarded Coulomb's law.