

Example Application: Supplemental Material

Here we provide one example of the type of problem we have developed and used with a Java application. This example considers the scattering of a plane electromagnetic wave from a thin conducting wire and was originally developed as part of a graduate course in electromagnetic theory. The section below is written at a level suitable for the instructor and so is more formal (and more terse) than appropriate for undergraduates. As needed such advanced description will be provided with the *Companion* so that the instructor will have full access to the details of formalism and the techniques used.

This and other samples will be made available at <http://www.fermi.edu/ccli> as described in the main body of the proposal.

For students, a much more simplified explanation will be provided. The basic method presented below is outlined as

- A thin wire is modeled by a current only in the direction of the axis, uniformly distributed around the wire, i.e. independent of the usual ϕ angle as we move around the surface of the wire, but changing in the direction of the axis.
- The boundary conditions are that the total tangential electric field is zero on a perfect conductor.
- We divide the wire into short segments and write the current with an as yet undetermined coefficient that gives its magnitude and phase on each segment.
- The total tangential electric field should be zero. It is made up of the contributions from all the segments of the wire and the incident plane wave. We require these to give a zero field on average over the segment, which gives a linear equation for each segment.
- Solving the linear equations gives the current induced on each segment. From this current, we can calculate the scattered field, and therefore the cross section etc.

We follow the detailed explanation below with two example problems with solution outlines, to show how we use these in our teaching.

1 Thin Wire Scattering

Scattering of radar signals from wires dropped from airplanes is a concrete example of a problem with a history of experimental[1], theoretical[2], and practical[3] study. Since a wire is simply a thin conducting cylinder, we describe plane wave scattering from such a cylinder to illustrate the numerical solution of scattering from a conductor.

Our problem begins with a perfectly conducting circular cylinder of radius a and length d centered at the origin and aligned along the z axis as shown in Fig. 1. A plane electromagnetic wave with angular frequency ω propagates with a wave vector

\vec{k} at an angle θ_{inc} to the z axis. The scattered wave results from induced currents on the cylinder. The outgoing directions are specified using the usual spherical angles θ and ϕ . The incoming wave's polarization can be decomposed into a component in the $\hat{k} - \hat{z}$ plane, and a component orthogonal to that plane. We want to calculate the currents induced on the cylinder surface and the resulting scattered radiation.

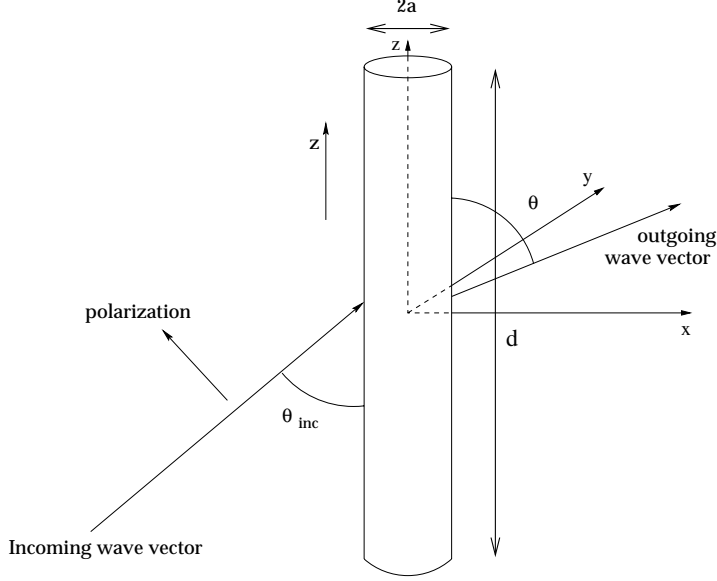


Figure 1: The geometry of the scattering problem

1.1 Model

We seek a solution valid for cylinders with a large aspect ratio, $a \ll d$, and incident wavelengths much larger than the radius, $\omega a \ll c$, but make no assumption about the relative size of the wavelength and the cylinder length.

The thin wire approximation provides a surface current only in the z -direction and uniformly distributed around the wire. Only the incident field polarized parallel to the wire induces such a surface current that is, therefore, of the form,

$$\vec{J}(\vec{r}) = \hat{z} \frac{I(z)\delta(\rho - a)}{2\pi a} \quad (1)$$

where $I(z)$ is the total current at a position z along the axis, $-d/2 < z < d/2$.

The Lorentz gauge vector potential from a harmonic current is given by

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \frac{\exp(ik|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} d^3 r'. \quad (2)$$

The electric field is given in terms of \vec{A} as

$$\vec{E}(\vec{r}) = ick\vec{A}(\vec{r}) + \frac{ic}{k} \vec{\nabla}[\vec{\nabla} \cdot \vec{A}(\vec{r})]. \quad (3)$$

On the surface of the wire the electric field has only a z-component,

$$E_z(\rho = a, z) = \frac{ic}{k} \left(\frac{\partial^2}{\partial z^2} + k^2 \right) A_z(\rho = a, z), \quad (4)$$

for which we require the vector potential at the surface,

$$A_z(\rho = a, z) = \frac{\mu_o}{4\pi} \int_{-d/2}^{d/2} I(z') K(z - z') dz',$$

where the kernel is

$$K(z - z') = \frac{1}{\pi} \int_0^\pi d\beta \frac{\exp \left(ik \sqrt{(z - z')^2 + 4a^2 \sin^2(\beta)} \right)}{\sqrt{(z - z')^2 + 4a^2 \sin^2(\beta)}}. \quad (5)$$

The incoming wave also contributes an electric field along the wire axis. We will approximate it by averaging its z component around the wire. (Since the wire is thin, this is very close to the value one would get by simply taking the value of the z -component of the incident field at the center of the wire.) The sum of this averaged incident electric field and the one from the current gives the total electric field which must be zero on the wire surface.

1.2 Numerical Solution Method

The solution can now be completed by defining a set of basis functions for the current. While the usual constant or piecewise linear basis functions will work, for variety (and because some of the integrations can be done analytically) we will choose the piecewise sinusoidal basis and test functions,

$$f_n(z) = \begin{cases} \frac{\sin(k|z - z_{n-1}|)}{\sin(k|z_n - z_{n-1}|)} & z_{n-1} < z < z_n \\ \frac{\sin(k|z - z_{n+1}|)}{\sin(k|z_n - z_{n+1}|)} & z_n < z < z_{n+1} \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

where n runs from 1 to N and

$$z_n \equiv \frac{nd}{N+1} - \frac{d}{2} \quad (7)$$

with z_0 and z_{N+1} being the ends of the cylinder. The basis functions and example fit are shown in Fig. 2. For large N this goes over to a piecewise linear basis.

The full basis functions are therefore

$$F_n(\vec{r}) = f_n(z) \frac{\delta(\rho - a)}{2\pi a}, \quad (8)$$

and we construct the current density on the wire surface as

$$\vec{J}(\vec{r}) = \hat{z} \sum_{n=1}^N I_n F_n(\vec{r}) \quad (9)$$

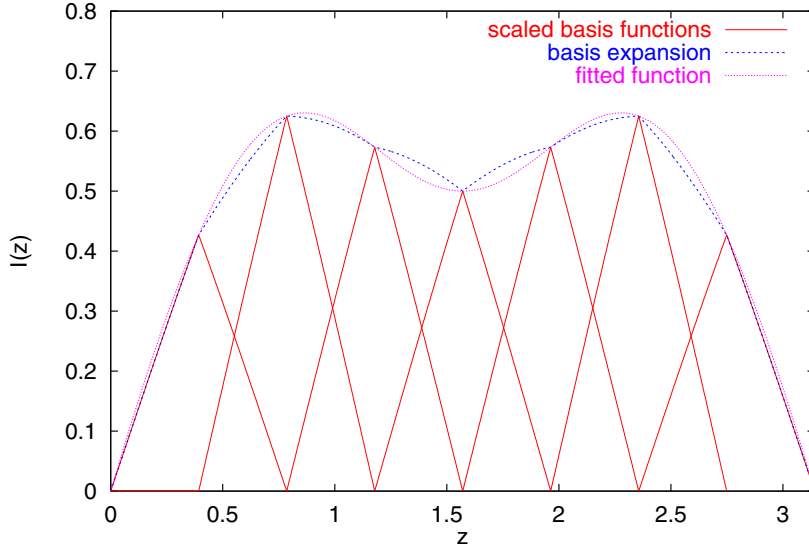


Figure 2: A current function and the corresponding basis functions multiplied by their coefficients as in Eq. 9

where I_n is the value of the current at z_n . It remains only to apply the boundary condition of zero electric field on the surface (Eq. 10) to produce N linear equations for the N unknowns, I_n .

The incident field (Fig. 1) with unit amplitude \hat{E}_0 , averaged with our test function, F_n , produces a field at the n -th segment that must exactly cancel the averaged field of the current,

$$\frac{\hat{E}_0 \cdot \hat{z}}{2\pi} \int d\phi \int_{z_{n-1}}^{z_{n+1}} dz f_n(z) e^{ikz \cos \theta_{inc}} e^{ika \cos \phi \sin \theta_{inc}} + \int_{z_{n-1}}^{z_{n+1}} dz f_n(z) E_z(\rho = a, z) = 0. \quad (10)$$

It is convenient to identify the first term in Eq. 10 as, $-V_n$, the integral of the incident field with the test function $F_n(\vec{r})$. This term cancels the average field from all currents, $\{I_m\}$ given by Eq. 4. Eq. 10 connects the segments' currents with $\{V_n\}$, and can be written as the N linear equations,

$$V_n = \sum_m Z_{nm} I_m. \quad (11)$$

The impedance matrix is obtained from the boundary condition and the use of Eq. 4,

$$Z_{nm} = \int_{z_{m-1}}^{z_{m+1}} dz \int_{z_{n-1}}^{z_{n+1}} dz' (ik f_m(z) f_n(z') + \frac{1}{ik} f'_m(z) f'_n(z')) K(z - z'). \quad (12)$$

Here $f'_m(z)$ is the derivative of $f_m(z)$ and the kernel is defined in Eq. 5.

The impedance matrix is a symmetric Toeplitz matrix. This linear system can be solved for the currents, $\{I_n\}$ by using a general complex linear equation solver, or by exploiting the Toeplitz structure. The software for this, including the Java user interface, is provided as described in Sec. 1.3.

The solution for the currents, $\{I_n\}$, permits calculation of the radiation fields at large r which are given by

$$\begin{aligned}\vec{H}_{sc}(\vec{r}) &= ik\hat{n} \times \vec{A}_{sc}(\vec{r})/\mu_o \\ \vec{E}_{sc}(\vec{r}) &= ikZ_o(\hat{n} \times \vec{A}_{sc}(\vec{r})) \times \hat{n}/\mu_o \\ \vec{A}_{sc}(\vec{r}) &= \frac{\mu_o}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{r}') e^{-ik\hat{n}\cdot\vec{r}'} d^3r',\end{aligned}$$

where \hat{n} points from the origin to the point of observation and $Z_o = \sqrt{\mu_o/\epsilon_o}$ is the free space impedance.

The power radiated by the scattered fields, r^2 times $\vec{S}_{sc} = Re(\vec{E}_{sc} \times \vec{H}_{sc}^*)/2$, and divided by the incident flux gives the cross section,

$$\frac{d\sigma(\theta)}{d\Omega} = \frac{k^2 Z_o^2}{2\mu_o^2} r^2 |A_z(\vec{r})|^2 \sin^2 \theta. \quad (13)$$

The vector potential at large r is given by

$$A_z(\vec{r}) = \frac{\mu_o}{4\pi} \frac{e^{ikr}}{r} J_o(ka \sin \theta) \sum_n \int_{z_{n-1}}^{z_{n+1}} dz f_n(z) e^{ikz \cos \theta}.$$

This integral is performed by the software and the differential cross section plotted as given in Sec. 1.3 below.

1.3 Results and User Interface

The WireScatter Java program has been provided to carry out the calculations for the model described in Sec. 1.2. This includes the graphical user interface (GUI) shown in Fig. 3 where input is provided for wire length, radius, incident plane wave direction and frequency. Output includes the angular distribution of radiation and the complex current distribution. These are provided graphically and can be saved to a data file. Further details of the use of the interface can be obtain by using the help menu of the GUI. The same calculations may be performed without the GUI using the **wire.f** Fortran and **WireScatterNoGui.java** Java codes.

1.4 Solved Problems

With the calculation of the current for arbitrary incident field being provided by the Java or Fortran codes, numerical calculation for all fields derived from the general expression of the vector potential, Eq. 2 is now possible. The results to the following questions are provided for illustration.

1. Obtain the back scatter differential cross section (i.e., for $\theta = \pi - \theta_{inc}$) multiplied by 4π . (This is called the monostatic radar cross section in the engineering literature.) Assume a 2370 MHz wave incident at an angle of $\pi/2$ and polarization parallel to a wire of 1/16 inch diameter. These values are chosen to match the measurements on steel wire of Chang and Liepa[1].

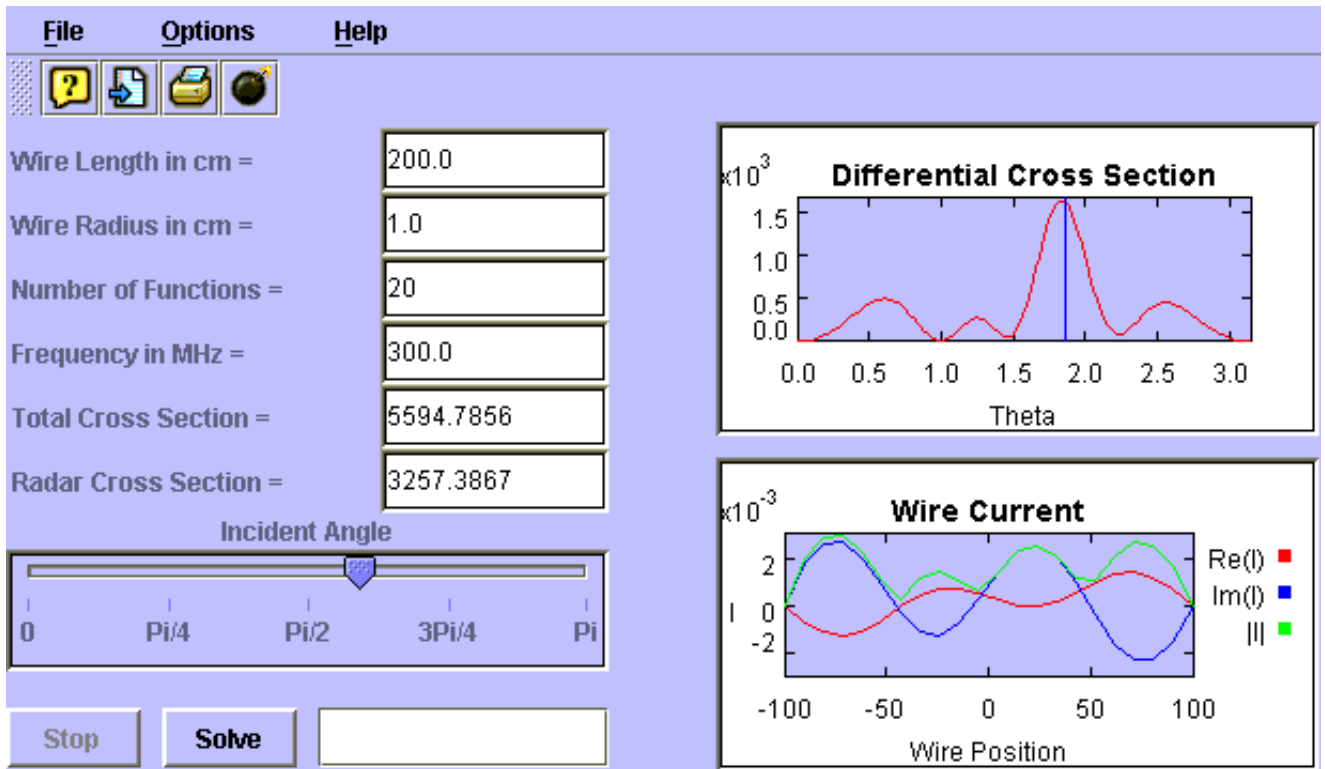


Figure 3: Screen Shot of the Java Program Interface for Wire Scattering

Compare your result with their experiment. Plot the dimensionless cross section, σ/λ^2 , as a function of the dimensionless wire length, d/λ .

Solution Outline

Fig. 4 shows the back scatter cross section obtained by our calculation and compares with some measurements of Chang and Liepa[1]. One sees the steps corresponding to resonance scattering when the current is approximately a cosine with a half integral number of wavelengths equal to the incident wave length.

2. The first peak in Fig. 4 corresponds to a half wave resonance in the back scatter cross section. For a 300 MHz incident wave, determine the length of a 1 cm radius wire necessary for resonance scattering that maximizes the total cross section at normal incidence. Compare the resultant current distribution to the cosine wave approximation.

Solution Outline

Using the *Companion* numerical interface, a resonance in total cross section is found at approximately $d = 44.5$ cm in a wire $a = 1$ cm, and $\theta_{inc} = \pi/2$. In Fig. 5 we show the magnitude of the current induced based on a calculation with these parameters and 2000 basis functions. The length is close to a half

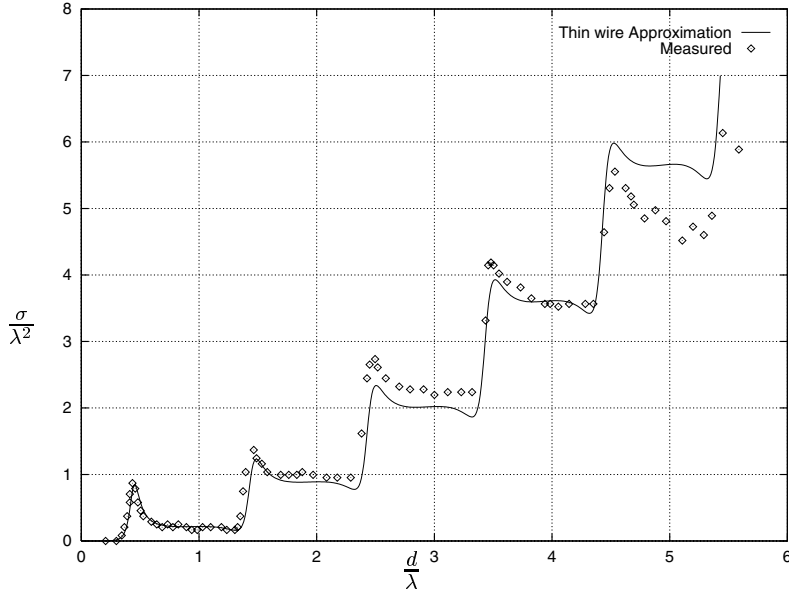


Figure 4: Back Scatter Cross Section of Steel Wire. (Measurements from Chang and Liepa.)

wave length, where we expect the wire to respond nearly resonantly to the incoming plane wave. This also maximizes the current and matches its phase with the incoming wave.

This result illustrates that in order to obtain resonance, one needs to use a wire shorter than the free space half wave length. The main reason for this is seen in the figure. For the bulk of the length, the current is well approximated by a sinusoidal function consistent with a sinusoidal approximations for antenna current. However, at the ends, the current drops much faster. Since the model of the wire does not include conducting circular end caps on the cylinder, we are really modeling a hollow thin tube. We know that in the electrostatics problem of an infinite plane the charge density diverges like $y^{-\frac{1}{2}}$ where y is the distance from the edge. We might suspect that a similar charge density exists for a tube if the radius of curvature is not too small. From the continuity equation, an oscillating charge density of this form would correspond to a current along z that goes like $y^{\frac{1}{2}}$. In the figure we also plot a square root function that closely matches the current near the ends.

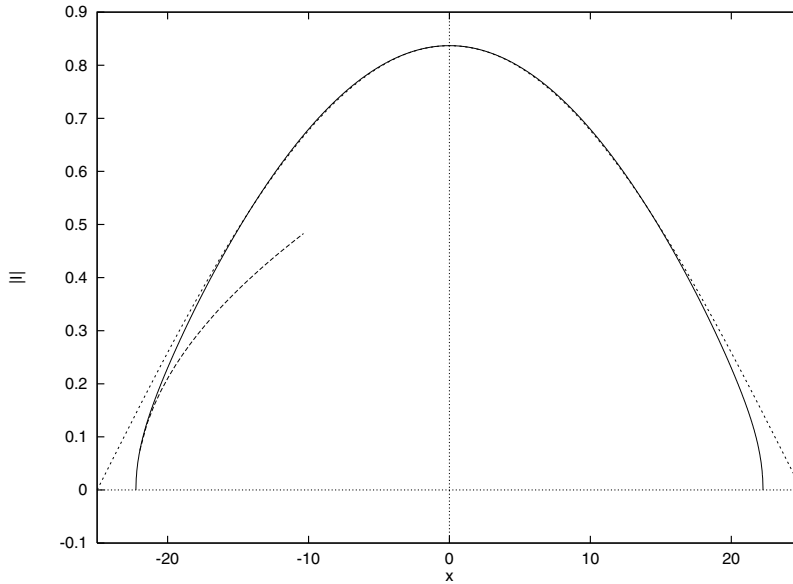


Figure 5: The magnitude of the current induced in a wire 44.5 cm long with radius 1 cm from a normally incident 300 MHz plane wave polarized along the wire (solid line). Also plotted are a cosine function with the free space wavelength (dotted line), and $0.14\sqrt{x + d/2}$ where d is the wire length (dashed line).

References

- [1] S. Chang and V. V. Liepa, *Measured Back Scattering Cross Section of Thin Wires*, University of Michigan Radiation Laboratory report No. 8077-4-T, May, 1967.
- [2] J. H. Van Vleck, F. Bloch, M. Hammermesh, *Theory of Radar Reflection from Wires or Thin Metallic Strips*, J. Appl. Phys. **18**, 274 (1947).
- [3] R. V. Jones, *The Wizard War, British Scientific Intelligence 1939-1945* (Coward, McCann & Geoghegan, New York, 1978).