Notes on the characteristic impedance of coax with a square outer conductor

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Abstract

These are some notes on calculating the characteristic impedance of a coaxial line with a square cross section outer conductor and a circular inner conductor. They were written in response to comments by Zack Lau, KH6CP, in May 1995 QEX that some handbooks had a "theoretically determined" formula for the characteristic impedance of coaxial cable with a square outer conductor that did not agree with the empirically determined result $138 log_{10}(1.08D/a)$ where D is the side of the square, and a is the diameter of the inner conductor. These are some notes that I sent Zack in September 1996 showing that transmission line theory predicts the 1.08 factor. Apparently, some handbooks suffered from a propagation of misprints.

The TEM mode characteristic impedance of coaxial lines is given by

$$Z_0 = \sqrt{L/C}, \tag{1}$$

where L and C are the inductance and capacitance per unit length. For an air dielectric and perfect cylindrical conductors, the waves travel at the speed of light so that L and C are related by

$$c = \frac{1}{\sqrt{LC}}, \tag{2}$$

which when combined with Eq. 1 shows that only the capacitance per unit length is needed to calculate the characteristic impedance,

$$Z_0 = \frac{1}{cC}. \tag{3}$$

One way to calculate the capacitance per unit length of a cylindrical structure is to realize that since there is no charge between the inner and outer conductors, the potential must be a solution of Laplace's equation

$$\nabla^2 \Phi = 0. \tag{4}$$

The capacitance can be calculated by solving Laplaces equation with the boundary condition that the potential is one volt on the outer conductor and zero volts on the inner conductor. The charge per unit length on either conductor is the capacitance per unit length.

For the coaxial case, the DC field does not change along the length of the coax. Laplaces equation therefore reduces to the two-dimensional equation,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0, \tag{5}$$

where x and y are cartesian coordinates of the cross sectional area. For a square cross section outer conductor, the potential must have the symmetry of the square. A general solution to Laplaces equation with that symmetry is,

$$\Phi = b_1 \ln(r) + \sum_{m=1} (b_{m+1} r^{4m} + a_{m+1} r^{-4m}) \cos(4m\phi), \tag{6}$$

where $\phi = \tan^{-1}(y/x)$, and $r^2 = x^2 + y^2$.

I take the case where the radius of the inner conductor is 1, and half the side length of the outer conductor is D. Enforcing the boundary condition of zero volts on the inner conductor makes $\Phi = 0$ at $r = 1$, which means $a_i = -b_i$. The solution for Φ is then

$$\Phi = b_1 \ln(r) + \sum_{m=1} b_{m+1} \cos(4m\phi)(r^{4m} - r^{-4m}). \tag{7}$$

I must now enforce the condition that $\Phi = 1$ on the outer conductor. I do this by point matching. I simply take points evenly spaced in the angle ϕ and require that Φ be one on the outer conductor at these angles. This gives a set of linear equations for the coefficients b_i that can be solved. Specifically for N coefficents, I require that

$$b_1 \ln(r_i) + \sum_{m=1}^{N-1} b_{m+1} \cos(4m\phi_i)(r_i^{4m} - r_i^{-4m}) = 1, \tag{8}$$

for all i values 1 to N, and

$$\phi_i = \frac{(2i-1)\pi}{8N}, \qquad r_i = \frac{D}{\cos(\phi_i)}. \tag{9}$$

Once the b_i values are solved, I need to calculate the charge per unit length. The charge density on a conductor is the normal electric field times the dielectric constant. In our case this is ϵ_0. The normal electric field is most easily calculated for the inner conductor. Integrating around the circular inner conductor immediately gives zero contribution for all but the b_1 term. The b_1 term give a charge per unit length of

$$q = 2\pi\epsilon_0 b_1, \tag{10}$$

and the characteristic impedance is

$$Z_0 = \frac{1}{2\pi\epsilon_0 c b_1} = \frac{60}{b_1}. \tag{11}$$

Before looking at the numerical results for larger N, I can solve analytically for the case of $N = 1$. In that case I have a purely logarithmic potential and match the potential to 1 at the point given by $\phi = \pi/8$. Eq. 8 becomes,

$$b_1 \ln\left(\frac{D}{\cos(\pi/8)}\right) = 1, \tag{12}$$

and evaluating

$$\frac{1}{\cos(\pi/8)} = 1.08, \tag{13}$$

the characteristic impedance is approximately

$$Z_0 = 60 \ln(1.08\ D), \tag{14}$$

in agreement with the empirical value. The results for larger values of N are easily calculated on the computer. One way to write the results is as

$$Z_0 = 60 \ln(\alpha(D)\ D). \tag{15}$$

Z_0 can only depend on the ratio of D/a where D is the outer conductor halfside and a is the inner conductor radius. This ratio is the same as the outer conductor side divided by the inner conductor diameter, so that can be substituted as well. Substituting D/a for D gives the final result.

Table **??** shows the calculated α and Z_0 with N=10, for D/a from 1.1 to 6.0. The value starts at 1.06 at $D/a = 1.1$ where $Z_0 = 9$ Ohms. α becomes 1.08 for $D/a = 1.275$ where $Z_0 = 19$ Ohms. It remains at 1.08 thereafter. The asymptotic value of α is actually 1.0787. I have repeated the calculation with N=20, with no change in the results indicating good convergence.

The empirical value of 1.08 should work fine.

A final note for the compulsive nitpickers. The value 60 is really two times the numerical value of the speed of light times the appropriate power of ten. That is it is really 2×29.9792458 or 59.9584916.

The calculated characteristic impedance Z_0, for a coaxial air line with a square cross section outer conductor of side D and a circular cross section inner conductor of diameter a, as a function of D/a. The value of α where $Z_0 = 60\ln(\alpha D/a)$ is also shown.

$\frac{D}{a}$	α	Z_0	$\frac{D}{a}$	α	Z_0	$\frac{D}{a}$	α	Z_0
1.10000	1.06422	9.45326	2.40000	1.07868	57.07262	3.70000	1.07870	83.04562
1.12500	1.06724	10.97142	2.42500	1.07869	57.69448	3.72500	1.07870	83.44966
1.15000	1.06947	12.41565	2.45000	1.07869	58.30996	3.75000	1.07870	83.85100
1.17500	1.07118	13.80151	2.47500	1.07869	58.91918	3.77500	1.07870	84.24968
1.20000	1.07250	15.13895	2.50000	1.07869	59.52227	3.80000	1.07870	84.64572
1.22500	1.07355	16.43478	2.52500	1.07869	60.11936	3.82500	1.07870	85.03917
1.25000	1.07439	17.69392	2.55000	1.07869	60.71056	3.85000	1.07870	85.43005
1.27500	1.07507	18.92012	2.57500	1.07869	61.29598	3.87500	1.07870	85.81840
1.30000	1.07563	20.11629	2.60000	1.07869	61.87575	3.90000	1.07870	86.20425
1.32500	1.07609	21.28477	2.62500	1.07869	62.44996	3.92500	1.07870	86.58764
1.35000	1.07647	22.42752	2.65000	1.07870	63.01873	3.95000	1.07870	86.96860
1.37500	1.07679	23.54617	2.67500	1.07870	63.58215	3.97500	1.07870	87.34715
1.40000	1.07705	24.64213	2.70000	1.07870	64.14033	4.00000	1.07870	87.72333
1.42500	1.07728	25.71661	2.72500	1.07870	64.69336	4.02500	1.07870	88.09716
1.45000	1.07747	26.77070	2.75000	1.07870	65.24134	4.05000	1.07870	88.46868
1.47500	1.07763	27.80537	2.77500	1.07870	65.78436	4.07500	1.07870	88.83791
1.50000	1.07777	28.82147	2.80000	1.07870	66.32250	4.10000	1.07870	89.20489
1.52500	1.07789	29.81980	2.82500	1.07870	66.85587	4.12500	1.07870	89.56963
1.55000	1.07799	30.80108	2.85000	1.07870	67.38452	4.15000	1.07870	89.93217
1.57500	1.07808	31.76596	2.87500	1.07870	67.90857	4.17500	1.07870	90.29253
1.60000	1.07815	32.71506	2.90000	1.07870	68.42807	4.20000	1.07870	90.65074
1.62500	1.07822	33.64894	2.92500	1.07870	68.94311	4.22500	1.07870	91.00683
1.65000	1.07827	34.56815	2.95000	1.07870	69.45377	4.25000	1.07870	91.36081
1.67500	1.07832	35.47317	2.97500	1.07870	69.96012	4.27500	1.07870	91.71272
1.70000	1.07837	36.36448	3.00000	1.07870	70.46222	4.30000	1.07871	92.06258
1.72500	1.07840	37.24250	3.02500	1.07870	70.96016	4.32500	1.07871	92.41040
1.75000	1.07844	38.10766	3.05000	1.07870	71.45401	4.35000	1.07871	92.75623
1.77500	1.07847	38.96036	3.07500	1.07870	71.94382	4.37500	1.07871	93.10007
1.80000	1.07849	39.80095	3.10000	1.07870	72.42966	4.40000	1.07871	93.44195
1.82500	1.07851	40.62980	3.12500	1.07870	72.91160	4.42500	1.07871	93.78189
1.85000	1.07853	41.44725	3.15000	1.07870	73.38970	4.45000	1.07871	94.11992
1.87500	1.07855	42.25361	3.17500	1.07870	73.86402	4.47500	1.07871	94.45606
1.90000	1.07857	43.04919	3.20000	1.07870	74.33462	4.50000	1.07871	94.79032
1.92500	1.07858	43.83428	3.22500	1.07870	74.80155	4.52500	1.07871	95.12273
1.95000	1.07859	44.60917	3.25000	1.07870	75.26488	4.55000	1.07871	95.45331
1.97500	1.07860	45.37412	3.27500	1.07870	75.72466	4.57500	1.07871	95.78208
2.00000	1.07861	46.12939	3.30000	1.07870	76.18094	4.60000	1.07871	96.10906
2.02500	1.07862	46.87523	3.32500	1.07870	76.63378	4.62500	1.07871	96.43426
2.05000	1.07863	47.61187	3.35000	1.07870	77.08322	4.65000	1.07871	96.75771
2.07500	1.07864	48.33954	3.37500	1.07870	77.52933	4.67500	1.07871	97.07943
2.10000	1.07864	49.05846	3.40000	1.07870	77.97214	4.70000	1.07871	97.39943
2.12500	1.07865	49.76884	3.42500	1.07870	78.41171	4.72500	1.07871	97.71773
2.15000	1.07865	50.47089	3.45000	1.07870	78.84807	4.75000	1.07871	98.03436
2.17500	1.07866	51.16479	3.47500	1.07870	79.28129	4.77500	1.07871	98.34932
2.20000	1.07866	51.85074	3.50000	1.07870	79.71141	4.80000	1.07871	98.66264
2.22500	1.07867	52.52892	3.52500	1.07870	80.13846	4.82500	1.07871	98.97433
2.25000	1.07867	53.19950	3.55000	1.07870	80.56249	4.85000	1.07871	99.28440
2.27500	1.07867	53.86266	3.57500	1.07870	80.98355	4.87500	1.07871	99.59289
2.30000	1.07868	54.51856	3.60000	1.07870	81.40167	4.90000	1.07871	99.89979
2.32500	1.07868	55.16735	3.62500	1.07870	81.81690	4.92500	1.07871	100.20514
2.35000	1.07868	55.80920	3.65000	1.07870	82.22927	4.95000	1.07871	100.50894
2.37500	1.07868	56.44424	3.67500	1.07870	82.63883	4.97500	1.07871	100.81120
						5.00000	1.07871	101.11196